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A generalized Wigner function for quantum systems with the $SU(2)$ dynamical symmetry group

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Abstract
We introduce a Wigner-like quasidistribution function to describe quantum systems with the $SU(2)$ dynamic symmetry group. This function is defined in a three-dimensional group manifold and can be used to represent the states defined in several $SU(2)$ invariant subspaces. The explicit differential Moyal-like form of the star product is found and analyzed in the semiclassical limit.

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1. Introduction

The concept of phase-space representation of quantum mechanics, introduced by Wigner [1], provides a useful insight into a problem of correspondence between quantum and classical worlds. Numerous applications of the theory of phase-space methods to physical problems have been extensively discussed in the last decades [2, 3]. Several types of quasidistribution functions for physical systems with different types of dynamic symmetries were proposed, starting from the flat $q - p$ space [7, 8], spin-like systems [5, 11, 12] (see also [13–17]), finite-dimensional quantum systems [23], and recently introduced Wigner-like mapping for a wide class of continuous (Lie type) and discrete groups [18–21].

It turns out that the language of quasidistribution functions is very convenient not only for graphical representation of quantum states in the corresponding classical phase space, but also for analysis of quantum system's evolution in the semiclassical limit [25–34]. The dynamic properties of quasiclassical systems are usually studied in the framework of the Moyal correspondence, where both states and observables are considered as functions (Weyl symbols) on a phase space, in such a way that average values are computed by integration over the phase space of some quasi-distribution function with the Weyl symbol of a corresponding operator and the usual product of two operators is mapped onto the so-called star product of their symbols [4, 6]. Such a star product allows us to replace the standard manipulations of operators in the Hilbert space by a differential (or integral) operator acting on the product of Weyl symbols. Although, a general expression for the integral representation of the star product is easy to obtain (see, e.g., [19]), it is not useful to perform calculations (except the simplest case of the Heisenberg–Weyl group). In spite of its obvious convenience for practical purposes, the explicit differential form of the star product to the best of our knowledge is available only for quantum systems with the Heisenberg–Weyl [4, 9, 10] and $SU(2)$ [32] symmetry groups. An important advantage of the differential form of the star product is in the possibility of its expansion on the semiclassical parameter (which is different for each specific quantum system), which usually leads to an essential simplification of the evolution equation for the corresponding Wigner function (Weyl symbol of the density matrix). It has been shown that even such a truncated evolution equation describes quite well sufficiently long-time dynamics even for essentially nonlinear quantum systems [25, 26, 29, 30, 33].

In the present paper, we will be interested in the quasiclassical description of quantum systems with $SU(2)$ dynamic symmetry group. Usually, the Stratonovich–Weyl [5, 12, 16] quasidistribution function is used for the phase-space description of spin-like systems, which corresponds to some *fixed* finite-dimensional representation of the $SU(2)$ group. In this case, the classical phase space (which is the two-dimensional sphere $\mathcal{S}_2(\theta, \phi)$) with the corresponding symplectic structure is well defined and the Weyl mapping inside each irreducible subspace is invertible (one-to-one correspondence). Another, the so-called *metaphase-space* representation was introduced in [18], where the Wigner operator, which establishes the Weyl mapping, is defined as a Fourier transform of the $SU(2)$ group element in the polar parametrization (see also [24], where a very similar construction was used for representation of polarization states of light). Such Wigner-like functions, being invertible and covariant under group transformations, nevertheless do not have good analytical properties, related to the fact that the Fourier transform, in general, does not preserve the periodicity property of the group element. Besides, the polar parametrization is not convenient for obtaining the differential form of the star product. Recently, the 'midpoint' approach to the Wigner–Weyl mapping has been developed in [20, 21]. Nevertheless, such a very attractive and intuitive approach leads to a Wigner function which, in the particular case of the $SU(2)$ symmetry, depends on four discrete indices, besides the group parameters. As a result, only the integral form of the star product is obtained, which makes it not very suitable for the semiclassical analysis.

In this paper, we introduce an alternative Wigner-like quasidistribution function for quantum systems with $SU(2)$ dynamic symmetry group, which allows us to obtain a differential form of the star product and consequently analyze the quantum dynamics in the semiclassical limit. The principal difference with the standard Stratonovich mapping consists in taking into account simultaneously all the irreducible representations of the $SU(2)$ group over which the initial state of a quantum system is expanded, as well as to extend the star-product technology to the Hamiltonians not preserving $SU(2)$ invariant subspaces.

In section 2, we briefly recall the fundamentals of the Stratanovich–Weyl mapping for spin-like systems. In section 3, we define the Weyl mapping for the $SU(2)$ group in the Euler parametrization and discuss its main properties. In section 4, we derive the differential form of the star product and discuss its form in the semiclassical limit. The Hermiticity, covariance and some other properties are proved in the appendix A. The general expression for the star-product operator is obtained in the appendix B.

2. Stratonovich–Weyl correspondence for spin systems

According to the axiomatic approach to the phase-space formulation of quantum mechanics [5] we associate each operator $\hat{f}$ with its symbol $W_f(\Omega)$—a c-number function defined in the corresponding phase space. Obviously, such an (invertible) map $\hat{f} \rightarrow W_f(\Omega)$ depends on the ordering rules of functions of non-commutative operators, which is taken into account by introducing an additional index s, specifying a certain operator ordering, such that $\hat{f} \to W_f^{(s)}(\Omega)$, where the value $s = 0$ corresponds to the Stratonovich–Weyl symbol, while $s = \pm 1$ leads to the Beresin contravariant P-symbol and covariant Q-symbol, respectively. A general rule to associate with each operator $\hat{f}$ acting on a Hilbert space a function $W_f^{(s)}$ (s-parametrized symbol of $\hat{f}$) defined on the phase space is given by the 'Stratonovich–Weyl correspondence' [5, 16, 19]

$$W_f^{(s)}(\Omega) = \mathrm{Tr}(\hat{w}_s(\Omega)\hat{f}),$$

where $\hat{w}_s(\theta, \phi)$ is an s-parametrized Stratonovich–Weyl kernel. From now on we will be interested exclusively in the symmetric operator ordering, $s = 0$ (so that the symbol of the density matrix is commonly called the Wigner function), which possess adequate properties in the semiclassical limit [32].

The Stratonovich–Weyl kernel $\hat{w}(\theta, \phi)$ for spin-like systems (systems with $SU(2)$ dynamic symmetry group for a fixed $(2S + 1)$-dimensional irreducible representation) is introduced according to [5, 12, 16]

$$\hat{w}(\theta, \phi) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} Y_{LM}^*(\theta, \phi)\hat{T}_{LM}^S = \hat{w}^\dagger(\theta, \phi), \qquad (\theta, \phi) \in \mathcal{S}_2, \tag{1}$$

where $Y_{LM}(\theta, \phi)$ are the spherical harmonics, $\hat{T}_{LM}^S$ are the irreducible rank L tensor operators [38] which form an orthogonal operator basis in the space of $(2S + 1) \times (2S + 1)$ matrices and are defined as

$$\hat{T}_{LM}^S = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} C_{Sm,LM}^{Sm'} |S, m'\rangle\langle S, m|. \tag{2}$$

Here $C_{Sm,LM}^{Sm'}$ are the Clebsch–Gordan coefficients which couple two representations of spin S and L $(0 \leqslant L \leqslant 2S)$ to a total spin S. The kernel $\hat{w}(\theta, \phi)$ is normalized as

$$\mathrm{Tr}\,\hat{w}(\theta, \phi) = 1, \qquad \frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, \hat{w}(\theta, \phi) = I, \tag{3}$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi$ is the invariant measure on the sphere.

The Stratonovich–Weyl symbol of the operator $\hat{f}$,

$$W_f(\theta, \phi) = \mathrm{Tr}(\hat{f}\hat{w}(\theta, \phi)), \tag{4}$$

is covariant under rotations and provides the overlap relation

$$\frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, W_g(\theta, \phi) W_f(\theta, \phi) = \mathrm{Tr}(\hat{g}\hat{f}). \tag{5}$$

The operator $\hat{f}$ can be reconstructed from its symbol $W_f(\theta, \phi)$ (4) through the following relation:

$$\hat{f} = \frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, \hat{w}(\theta, \phi) W_f(\theta, \phi). \tag{6}$$

In terms of the expansion coefficients of an operator $\hat{f}$ from $(2S+1)$-dimensional representation of the universal enveloping algebra of $su(2)$ in the basis of irreducible tensor operators $\hat{T}_{lk}^S$ (2),

$$\hat{f} = \sum_{l=0}^{2S} \sum_{k=-l}^{l} f_{lk} \hat{T}_{lk}^S, \tag{7}$$

its Stratonovich–Weyl symbol takes the form

$$W_f(\theta,\phi) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \sum_{l,k} f_{lk} Y_{lk}(\theta,\phi). \tag{8}$$

The (associative but non-commutative) operation of star product reduces the calculation of the symbol of a product of two operators to an application of some operator $\hat{L}(\theta,\phi)$ to the product of individual symbols:

$$W_{fg} = W_f * W_g = \hat{L}(\theta,\phi)[W_f W_g]. \tag{9}$$

The star product allows us to rewrite the Schrödinger equation for the density matrix,

$$\mathrm{i}\partial_t \rho = [H,\rho],$$

as a Liouville-type evolution equation for the Wigner function,

$$\mathrm{i}\partial_t W_\rho = \{W_H, W_\rho\}_M, \tag{10}$$

where H is the system Hamiltonian and

$$\{W_f, W_g\}_M = W_f * W_g - W_g * W_f$$

is the so-called Moyal bracket.

The integral form of the star product immediately follows from definition (4) and the reconstruction relation (6)

$$W_{fg} = \left(\frac{2S+1}{4\pi}\right)^2 \int\int \mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\, K(\theta,\phi;\theta_1,\phi_1;\theta_2,\phi_2) W_f(\theta_1,\phi_1) W_g(\theta_2,\phi_2), \tag{11}$$

where

$$K(\theta,\phi;\theta_1,\phi_1;\theta_2,\phi_2) = \mathrm{Tr}\left[\hat{w}(\theta,\phi)\hat{w}(\theta_1,\phi_1)\hat{w}(\theta_2,\phi_2)\right]. \tag{12}$$

Unfortunately, this kernel has quite a complicated form and, thus, is not convenient for practical use. The differential form of the star product was found in [32] and possess as an important property that in the large spin limit, $\varepsilon = (2S+1)^{-1} \ll 1$, the Moyal brackets are reduced to the Poisson brackets in such a way that the first-order corrections disappear, so that the semiclassical evolution equation takes the form

$$\partial_t W_\rho \approx 2\varepsilon\{W_\rho, W_H\}_P + O(\varepsilon^3), \tag{13}$$

where $\{,\}_P$ denotes the Poisson brackets on the sphere

$$\{,\}_P = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\phi}_f \otimes \frac{\partial}{\partial\theta}_g - \frac{\partial}{\partial\theta}_f \otimes \frac{\partial}{\partial\phi}_g\right).$$

We stress, that only in the case of the Wigner function that terms of order ε^2 do not appear in (13) [32].

3. Generalized Wigner function

3.1. Definition and properties

In this section, we generalize the Stratonovich–Weyl kernel to the case of quantum systems for which the representation space is a direct sum of several $SU(2)$ irreducible subspaces.

Let us consider a quantum system with $SU(2)$ dynamic symmetry group, so that the allowed transformations are generated by elements from the $su(2)$ enveloping algebra. Given an arbitrary operator $\hat{f}$, it can be represented as a linear combination of the irreducible tensor operators $T_{Kq}^{J'J}$ as follows:

$$\hat{f} = \sum_{J,J'=0,1/2,1,\ldots}^{\infty} \sum_{K=|J'-J|}^{J'+J} \sum_{q=-K}^{K} f_{Kq}^{J'J} \hat{T}_{Kq}^{J'J}, \tag{14}$$

where

$$\hat{T}_{Kq}^{J'J} = \sum_{mm'} \sqrt{\frac{2K+1}{2J'+1}} C_{JmKq}^{J'm'} |J', m'\rangle\langle J, m|, \tag{15}$$

here $C_{JmKq}^{J'm'}$ are the Clebsch–Gordan coefficients which couple two representations of spin J and K ($|J'-J| \leqslant K \leqslant J+J'$) to a total spin J'. The tensors (15) form an orthonormal basis

$$\mathrm{Tr}\left(T_{K_1 q_1}^{J_1' J_1} T_{K_2 q_2}^{\dagger J_2' J_2}\right) = \delta_{J_1' J_2'} \delta_{J_1 J_2} \delta_{K_1 K_2} \delta_{q_1 q_2}, \qquad T_{Kq}^{\dagger J'J} = (-1)^{2K+J'-J+q} T_{K-q}^{JJ'}, \tag{16}$$

so that the expansion coefficients $f_{KQ}^{J'J}$ are

$$f_{KQ}^{J'J} = \mathrm{Tr}\big(T_{KQ}^{\dagger J'J} \hat{f}\big).$$

The tensors $T_{kl}^{J'J}$ possess the following transformation property under the $SU(2)$ rotations:

$$T_g T_{Kl}^{J'J} T_g^{\dagger} = \sum_q T_{Kq}^{J'J} D_{ql}^{K}(\phi, \theta, \psi), \tag{17}$$

where

$$T_g = \mathrm{e}^{-\mathrm{i}\phi S_z} \mathrm{e}^{-\mathrm{i}\theta S_y} \mathrm{e}^{-\mathrm{i}\psi S_z}, \qquad 0 \leqslant \phi < 2\pi, \qquad 0 \leqslant \theta < \pi, \qquad 0 \leqslant \psi < 4\pi,$$

and $D_{ql}^{k}(\phi, \theta, \psi)$ is the Wigner D-function in Euler parametrization, satisfying the orthogonality relation

$$\frac{2k_1+1}{16\pi^2} \int \mathrm{d}V\, D_{q_1 q_1'}^{k_1 *}(\phi, \theta, \psi)\, D_{q_2 q_2'}^{k_2}(\phi, \theta, \psi) = \delta_{k_1 k_2} \delta_{q_1 q_2} \delta_{q_1' q_2'}, \qquad \mathrm{d}V = \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}\psi.$$

Changing the summation indices in (14), $j = J' + J$, $Q' = J' - J$ and making use of the resummation formula

$$\sum_{Q'=-j}^{j} \sum_{K=|Q'|}^{j} a_{Q'K} = \sum_{K=\{0,1/2\}}^{j} \sum_{Q'=-K}^{K} a_{Q'K},$$

where $\{0, 1/2\}$ means that the index K runs from 0 or $1/2$ for integer and semi-integer values of j respectively, we obtain the following expansion:

$$\hat{f} = \sum_{j=0,1/2,1,\ldots}^{\infty} \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K} f_{KQ}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} T_{KQ}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} = \sum_{j=0,1/2,1,\ldots}^{\infty} \hat{f}_j. \tag{18}$$

Now we define the kernel operator as follows:

$$\hat{\omega}_j(\Theta) = \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K} \sqrt{\frac{2K+1}{j+1}} D_{QQ'}^{K}(\Theta) T_{KQ}^{\frac{j+Q'}{2}\frac{j-Q'}{2}}, \tag{19}$$

where $\Theta = (\phi, \theta, \psi)$. Using the transformation property (17) we can represent the kernel (19) in the following symmetric form:

$$\hat{\omega}_j(\Theta) = T_g(\Theta) P_j T_g^{\dagger}(\Theta),$$

where

$$P_j = \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K} \sqrt{\frac{2K+1}{j+1}} T_{KQ}^{\frac{j+Q'}{2}\,\frac{j-Q'}{2}}.$$

So that, the j-symbol of an operator $\hat{f}$ is defined as

$$W_f^j(\Theta) = \mathrm{Tr}(\hat{f}\hat{\omega}_j(\Theta)). \tag{20}$$

Using expansion (18) and the orthogonality relation (16) we easily find

$$W_f^j(\Theta) = \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K} \sqrt{\frac{2K+1}{j+1}} f_{KQ}^{\frac{j+Q'}{2}\,\frac{j-Q'}{2}} D_{QQ'}^{K*}(\Theta). \tag{21}$$

Note, that if $\hat{f}$ is an operator acting in a single $SU(2)$ irreducible subspace, then automatically $Q' = 0$ in the above equation and we reconstruct the standard Stratonovich–Weyl symbol (independent of the angle ψ) in the irreducible subspace of dimension $j + 1$.

The above kernel possesses the following properties (proved in appendix A):

(a) Hermiticity

$$\hat{\omega}_j^\dagger(\Theta) = \hat{\omega}_j(\Theta); \tag{22}$$

(b) Covariance

$$T_g\hat{\omega}_j(\Theta)T_g^\dagger = \hat{\omega}_j(g\cdot\Theta), \tag{23}$$

where $g\cdot\Theta$ means the standard transformation of Euler angle under $SU(2)$ group rotations [38];

(c) Normalization

$$\frac{j+1}{16\pi^2}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \int_0^{4\pi} \mathrm{d}\psi\,\hat{\omega}_j(\Theta) = \begin{cases} I_{j+1}, & j \text{ integer} \\ 0, & j \text{ semi-integer;} \end{cases} \tag{24}$$

(d) The trace condition

$$\mathrm{Tr}\,\hat{\omega}_j(\Theta) = \begin{cases} 1, & j \text{ integer} \\ 0, & j \text{ semi-integer} \end{cases}; \tag{25}$$

(e) Reproductive kernel

$$\Delta_{jj'}(\Theta;\Theta') = \mathrm{Tr}\big(\hat{\omega}_j(\Theta)\hat{\omega}_{j'}^\dagger(\Theta')\big) = \delta_{jj'}\Delta_j(\Theta;\Theta'), \tag{26}$$

where $\Delta_j(\Theta;\Theta')$ is an analog of the delta function on the group manifold, when it is acting on the j-symbols, in the sense that

$$\frac{j+1}{16\pi^2}\int \mathrm{d}V'\Delta_j(\Theta;\Theta')W_f^j(\Theta') = W_f^j(\Theta). \tag{27}$$

Equation (26) immediately leads to the reconstruction relation

$$\hat{f} = \sum_{j=0,1/2,1,\ldots}^{\infty} \hat{f}_j, \tag{28}$$

where the j-component of the operator $\hat{f}$ (compare with (18)) is expressed as

$$\hat{f}_j = \frac{j+1}{16\pi^2}\int \mathrm{d}V\, W_f^j(\Theta)\hat{\omega}_j(\Theta), \tag{29}$$

and the overlap relation takes the following form:

$$\mathrm{Tr}(\hat{f}\hat{g}) = \sum_{j=0,1/2,1,\ldots}^{\infty} \frac{j+1}{16\pi^2} \int \mathrm{d}V\, W_f^j(\Theta) W_g^j(\Theta). \tag{30}$$

It is worth noting that because the symbol of the identity operator (in the whole space) is

$$W_I^j(\Theta) = \sum_{n=0,1,2,\ldots}^{\infty} \delta_{jn},$$

the normalization condition is

$$\mathrm{Tr}\,\hat{f} = \sum_{j=0,1,2,\ldots}^{\infty} \frac{j+1}{16\pi^2} \int \mathrm{d}V\, W_f^j(\Theta).$$

Finally, the kernel (19) satisfies all the Stratonovich–Weyl postulates and can be used for (an invertible) mapping of operators into c-number functions. Note, that we do not fix the dimension of group representation, which reflects in the dependence of the kernel, and consequently the j-symbols, on three angles. In this sense, the mapping (21) is not a representation of operators in the classical phase space (which necessarily should be of even dimension). The main difference with the standard Stratonovich–Weyl mapping (see section 2) is in the possibility of reconstruction, equations (28) and (29), of the whole operator (in particular, the density matrix) and not only its projection on irreducible subspaces. Nevertheless, the j-symbol of any operator acting in a single $SU(2)$-irreducible subspace does not depend on the angle ψ (due to $Q' = 0$) and has the standard Stratonovich–Weyl form (8). It is worth noting that the index j takes only integer values for the symbols which do not depend on the angle ψ.

On the other hand, the standard definition (1) of the kernel (for a single irreducible representation of the $SU(2)$ group) is obtained by integrating $\hat{\omega}_j(\Theta)$ over ψ,

$$\int_0^{4\pi} \frac{\mathrm{d}\psi}{4\pi} \hat{\omega}_j(\Theta) = \hat{w}_{j/2}(\theta, \phi),$$

where $\hat{w}(\theta, \phi)$ is the Stratonovich–Weyl kernel (1).

For representation purposes sometimes it is convenient to use the polar parametrization $(\omega, \vartheta, \varphi)$ of the j-symbols instead of the Euler angles. Then, each j-symbol can be visualized as a function on S_3, where ω, $0 \leqslant \omega < 4\pi$ represents the radius of the three-dimensional sphere (a meta-phase space [18]). We note that the j-symbol is explicitly periodic on ω (compare with [24]).

3.2. Examples

Let us consider some examples of application of the map (20) and (21). First of all we note that the j-symbols of the $su(2)$ algebra generators have the standard Stratonovich–Weyl form [16, 32]

$$W_{S_k}^j(\Theta) = \sqrt{j/2(j/2+1)}\, n_k \sum_{n=0,1,\ldots} \delta_{jn}, \tag{31}$$

where δ-functions indicate admissible values of the index j and n_k are components of the unitary vector $\vec{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. In the same way, the symbols of squares of generators and the total angular momentum operator $J^2 = S_x^2 + S_y^2 + S_z^2$ are

$$W_{S_k^2}^j(\theta, \phi) = \frac{1}{4}\left(\sqrt{(j+3)(j-1)j(j+2)}\left(n_k^2 - \frac{1}{3}\right) + \frac{j(j+2)}{3}\right) \sum_{n=0,1,\ldots} \delta_{j,n}. \tag{32}$$

$$W_{J^2}^{j}(\Theta) = j/2(j/2+1)\sum_{n=0,1,\ldots}\delta_{jn}. \tag{33}$$

The j-symbol of the $\hat{z}$ operator, defined by its action on the spherical harmonics $\hat{z}Y_{jm}(\vartheta,\varphi) = \cos\vartheta Y_{jm}(\vartheta,\varphi)$, depends on the angle ψ,

$$W_{z}^{j}(\Theta) = \sin\theta\cos\psi\sum_{n=0,1,\ldots}\delta_{j,2n+1}, \tag{34}$$

due to the operator $\hat{z}$ mixes different $SU(2)$ irreducible subspaces. One can observe that only odd values of j are admissible in this case.

As a second nontrivial example we find the Wigner j-function (symbol of the density matrix) corresponding to the Bell state $|\Psi\rangle = (|0,0\rangle + |1,1\rangle)/\sqrt{2}$,

$$W_{\Psi}^{j}(\Theta) = \frac{1}{2}\left[\delta_{0,j} + \sqrt{\frac{3}{2}}(1+\cos\theta)\cos(\phi+\psi)\delta_{1,j} + \left(\frac{1}{3} - \frac{\cos\theta}{\sqrt{2}} + \sqrt{\frac{5}{2}}\frac{3\cos^2\theta - 1}{6}\right)\delta_{2,j}\right].$$

As expected the Wigner function 'lives' in the subspaces with $j = 0, 1, 2$. It is worth noting here that the term $\sim\delta_{1,j}$ appears as an image of the operator $|1,1\rangle\langle 0,0| + |0,0\rangle\langle 1,1|$, which is non-diagonal on the representations of the $SU(2)$ group.

Finally, we will find the symbol of the two-mode coherent state

$$|\alpha\beta\rangle = |\alpha\rangle|\beta\rangle = \mathrm{e}^{-\frac{|\alpha|^2+|\beta|^2}{2}}\sum_{n,m=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!m!}}|n\rangle|m\rangle. \tag{35}$$

The state $|\alpha\beta\rangle$ depends on three real parameters (apart from the overall global phase), thus, using the parameterization

$$|\alpha\beta\rangle = \sum_{N=0,1/2,1,\ldots}\mathrm{e}^{-\mathrm{i}N\psi_0}q_N|N;\theta_0,\varphi_0\rangle, \qquad r^2 = |\alpha|^2 + |\beta|^2, \qquad q_N = \mathrm{e}^{-r^2/2}\frac{r^{2N}}{\sqrt{(2N)!}},$$

$$\alpha = \mathrm{e}^{-\mathrm{i}(\varphi_0+\psi_0)/2}r\cos\theta_0/2, \qquad \beta = \mathrm{e}^{-\mathrm{i}(\psi_0-\varphi_0)/2}r\sin\theta_0/2,$$

$$|N;\theta_0,\varphi_0\rangle = \sum_{k=-N}^{N}\sqrt{\frac{(2N)!}{(N-k)!(N+k)!}}\,\mathrm{e}^{-\mathrm{i}k\varphi_0}\sin^{N-k}\theta_0/2\cos^{N+k}\theta_0/2|k,N\rangle,$$

we obtain the following Wigner j-function:

$$W_{\alpha\beta}^{j}(\Theta) = \frac{r^{2j}\,\mathrm{e}^{-r^2}}{\sqrt{j+1}}\sum_{K=\{0,1/2\}}^{j}\frac{2K+1}{\sqrt{(j+K+1)!(j-K)!}}\chi^{K}(\omega), \tag{36}$$

where $\chi^K(\omega)$ is the $SU(2)$ group character

$$\chi^{K}(\omega) = \frac{\sin\left[(2K+1)\frac{\omega}{2}\right]}{\sin\frac{\omega}{2}},$$

and

$$\cos\frac{\omega}{2} = \cos\frac{\theta-\theta_0}{2}\cos\frac{\phi-\phi_0}{2}\cos\frac{\psi-\psi_0}{2} - \cos\frac{\theta+\theta_0}{2}\sin\frac{\phi-\phi_0}{2}\sin\frac{\psi-\psi_0}{2}.$$

For large values of r, we have $j\sim r\gg 1$, then the sum (36) can be estimated as follows:

$$W_{\alpha\beta}^{j}(\Theta) \approx \frac{r^{2j}\,\mathrm{e}^{-r^2}}{\Gamma(j+2)}\sum_{K=\{0,1/2\}}^{j}(2K+1)\chi^{K}(\omega)$$

$$= \frac{r^{2j}\,\mathrm{e}^{-r^2}}{\Gamma(j+2)}\frac{2}{\sin\frac{\omega}{2}}\partial_\omega\sin\frac{\omega}{2}\begin{cases}(\chi^{j/2}(\omega))^2, & \text{integer } j\\ \chi^{(j+1/2)/2}(\omega)\chi^{(j-1/2)/2}(\omega), & \text{semi-integer } j\end{cases}.$$

Observe that the polar parametrization is especially suitable in this case, because the Wigner function depends exclusively on the polar angle ω.

4. The star-product operator

4.1. General expression

The star-product operator, L^j_{fg},

$$W^j_{fg}(\Theta) = L^j_{fg}\big(W^j_f(\Theta)W^j_g(\Theta)\big),$$

is derived in appendix B and has the following explicit form:

$$L^j_{fg} = \int_0^{4\pi} \frac{d\varphi\, d\varphi'}{(4\pi)^2} \sum_{n=0}^{\infty} \sum_{j_1, j_2=0,1/2,\ldots}^{\infty} a^n_{\frac{j_1+j_2-j}{2}} \sqrt{\frac{(j_1+1)(j_2+1)}{j+1}} F_j^{-1}(J^2) \tag{37}$$

$$\times \left[\big((J^+)^n F_{j_1}(J^2)\, e^{i(j_2-j+J^0)\varphi'}\big) \otimes \big((J^-)^n F_{j_2}(J^2)\, e^{i(j_1-j-J^0)\varphi}\big)\right], \tag{38}$$

where

$$J^{\pm} = i\, e^{\mp i\psi}\left[\pm \cot\theta \frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\theta} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right], \qquad J^0 = -i\frac{\partial}{\partial\psi},$$

are the contravariant components of the $SU(2)$ group generators and

$$J^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right]$$

is the corresponding Casimir operator, $[J^{\pm,0}, J^2] = 0$, in the rotating frame [38]

$$a^n_J = \frac{(-1)^n}{n!(2J+n+1)!},$$

and $F_j(J^2)$ is the operator-valued function, defined by its action on the Wigner D-function, $F_j(J^2)D^k_{nm}(\Theta) = \sqrt{(k+j+1)!(j-k)!}D^k_{nm}(\Theta)$. The integrals on φ and φ' can be substituted by the operator δ-functions, $\delta(j_1 - j - J^0)$ and $\delta(j_2 - j + J^0)$, and the sum on n can be formally evaluated

$$L^j_{fg} = \sum_{j_1, j_2=0,1/2,\ldots}^{\infty} \sqrt{\frac{(j_1+1)(j_2+1)}{j+1}} F_j^{-1}(J^2)\sigma\big(J_f^+ \otimes J_g^-\big)$$
$$\times \left[\big(F_{j_1}(J^2)\delta(j_2-j+J^0)\big) \otimes \big(F_{j_2}(J^2)\delta(j_1-j-J^0)\big)\right],$$

where

$$\sigma(z) = \sum_n \frac{(-z)^n}{n!(j_1+j_2-j+n+1)!} = \frac{1}{(\sqrt{z})^{j_1+j_2-j+1}} J_{j_1+j_2-j+1}(2\sqrt{z})$$

and $J_n(x)$ is the Bessel function. Nevertheless, the form (37) is more convenient for practical applications.

Note, that if both operators act in a single $SU(2)$ irreducible subspace (so that $\partial_\psi W_f(\Theta) = \partial_\psi W_g(\Theta) = 0$), the integration over φ and φ' can be immediately performed and the known form [32] for the standard Stratonovich–Weyl star product is automatically restored.

The star-product operator (37) allows us to rewrite the evolution equation for the density matrix in the Liouville-like (10) form for the j-symbol of the density matrix $W^j(\Theta) \equiv W^j_\rho(\Theta)$,

$$i\partial_t W^j(\Theta) = \big(L^j_{H\rho} - L^j_{\rho H}\big)\big(W^j_H(\Theta)W^j(\Theta)\big), \tag{39}$$

where $W^j_H(\Theta)$ is the j-symbol of the Hamiltonian.

4.2. Large j-limit

In the limit $j \gg 1$, which corresponds to the physical situation when the initial state is distributed among several $SU(2)$ irreducible subspaces of large dimensions, the general expression (37) can be essentially simplified. First of all, we rewrite (37) in the following form:

$$L_{fg}^{j} = \int_0^{4\pi} \frac{\mathrm{d}\varphi\,\mathrm{d}\varphi'}{(4\pi)^2} \sum_{n=0}^{\infty} \sum_{j_1,j_2=0,1/2,\ldots}^{\infty} a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}} \sqrt{\frac{(j_1+1)(j_2+1)}{j+1}} F_j^{-1}(J^2)$$
$$\times \left[\left((J^+)^n F_{j_1}(J^2)\,\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'}\right) \otimes \left((J^-)^n F_{j_2}(J^2)\,\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi}\right)\right], \tag{40}$$

where explicitly

$$a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}} = \frac{(-1)^n}{n!\Gamma(j+1+n-J^0\otimes I+I\otimes J^0+1)}.$$

Making use of the asymptotic relation $\Gamma(j+b+1) \simeq j^b\Gamma(j+1)$, so that

$$a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}} \approx (-1)^n \frac{(j+1)^{J^0\otimes I-I\otimes J^0-n}}{n!(j+1)!},$$
$$\frac{(j_1+1)!}{(j+1)!} \approx (j+1)^{I\otimes J^0}, \qquad \frac{(j_2+1)!}{(j+1)!} \approx (j+1)^{-J^0\otimes I},$$

we can perform summation on n in (40) obtaining

$$\frac{(j_1+1)!}{(j+1)!} \sum_{n=0}^{\infty} (J^+)^n \otimes (J^-)^n a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}} \approx \frac{1}{(j_2+1)!} \exp\left[-\frac{J^+\otimes J^-}{j+1}\right]. \tag{41}$$

Now, we represent the function $F_j(k)$ in the following manner:

$$F_j(k) = \frac{(j+1)!}{\sqrt{j+1}} \left[\prod_{n=0}^{k} \frac{1+\varepsilon n}{1-\varepsilon n}\right]^{\frac{1}{2}} = \frac{(j+1)!}{\sqrt{j+1}} \exp\left[\frac{1}{2}\sum_{n=0}^{k} \ln\frac{1+\varepsilon n}{1-\varepsilon n}\right],$$

where $\varepsilon = (j+1)^{-1}$. Expanding the logarithm

$$\sum_{n=0}^{k} \ln\frac{1+\varepsilon n}{1-\varepsilon n} = \sum_{n=0}^{k}\sum_{m=0}^{\infty} \frac{2(\varepsilon n)^{2m+1}}{2m+1}$$
$$= 2\sum_{n=0}^{k}\left[\varepsilon n + \frac{\varepsilon^3}{5}n^3 + \cdots\right] = \varepsilon k(k+1) + \frac{\varepsilon^3}{10}[k(k+1)]^2 + \cdots,$$

we obtain the following approximate expression:

$$F_j(J^2) \simeq (j+1)!\sqrt{\varepsilon}\exp\left(\frac{J^2}{2}\varepsilon\right). \tag{42}$$

Substituting (41) and (42) into (40) we approximate the L_{fg}^{j} operator as follows:

$$L_{fg}^{j} \simeq \exp\left(-\frac{J^2}{2}\varepsilon\right)\exp[-\varepsilon J^+\otimes J^-]\left[\exp\left(\frac{J^2}{2}\varepsilon\right)\otimes\exp\left(\frac{J^2}{2}\varepsilon\right)\right] \tag{43}$$

$$\times \int_0^{4\pi} \frac{\mathrm{d}\varphi\,\mathrm{d}\varphi'}{(4\pi)^2} \sum_{j_1,j_2=0,1,\ldots}^{\infty} (\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'}) \otimes (\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi}), \tag{44}$$

where the first operator, $\exp(-\varepsilon J^2/2)$, acts on both symbols.

Finally, applying the Baker–Hausdorff expansion

$$\mathrm{e}^{\hat f}\,\mathrm{e}^{\hat g} = \mathrm{e}^{\hat f+\hat g+\frac{1}{2}[\hat f,\hat g]+\cdots},$$

and using the identity

$$J^2(fg) = (J^2\otimes I + I\otimes J^2 + 2J^0\otimes J^0 - J^+\otimes J^- - J^-\otimes J^+)(fg),$$

we arrive at the following approximate expression for the star-product operator:

$$L^j_{fg} \simeq \exp\left[-\varepsilon\frac{2J^0\otimes J^0 + J^+\otimes J^- - J^-\otimes J^+}{2}\right] \times \int_0^{4\pi}\frac{\mathrm{d}\varphi'\,\mathrm{d}\varphi}{(4\pi)^2}\sum_{j_1,j_2=0,1/2,\ldots}^{\infty}(\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi})\otimes(\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi'}),$$

where the first operator exponent can still be expanded further in series,

$$\exp\left[-\varepsilon\frac{2J^0\otimes J^0 + J^+\otimes J^- - J^-\otimes J^+}{2}\right] \approx I - \varepsilon J^0\otimes J^0 - \frac{\varepsilon}{2}(J^+\otimes J^- - J^-\otimes J^+),$$

in our order of exactitude.

Thus, the large j-limit of the evolution equation (39) acquires the form

$$\begin{aligned}\mathrm{i}\partial_t W^j(\Theta) = {}& \frac{\varepsilon}{2}(J^-\otimes J^+ - J^+\otimes J^-)\int_0^{4\pi}\frac{\mathrm{d}\varphi\,\mathrm{d}\varphi'}{(4\pi)^2}\\ &\times\sum_{j_1,j_2}^{\infty}\big[\big(\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'}W_H^{j_1}(\Theta)\big)(\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi}W^{j_2}(\Theta))\\ &+(\mathrm{e}^{\mathrm{i}(j_2-j-J^0)\varphi'}W_H^{j_1}(\Theta))(\mathrm{e}^{\mathrm{i}(j_1-j+J^0)\varphi}W^{j_2}(\Theta))\big]\\ &+(I-\varepsilon J^0\otimes J^0)\int_0^{4\pi}\frac{\mathrm{d}\varphi\,\mathrm{d}\varphi'}{(4\pi)^2}\sum_{j_1,j_2}^{\infty}\big[(\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'}W_H^{j_1}(\Theta))(\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi}W^{j_2}(\Theta))\\ &-\big(\mathrm{e}^{\mathrm{i}(j_2-j-J^0)\varphi'}W_H^{j_1}(\Theta)\big)(\mathrm{e}^{\mathrm{i}(j_1-j+J^0)\varphi}W^{j_2}(\Theta))\big],\end{aligned}$$

where j_1, j_2 take integer and semi-integer values. It is clear that in the case when both symbols $W^j(\Theta)$ and $W_H^j(\Theta)$ do not depend on the angle ψ, the last two terms in the above equation take the same value and thus, are cancelled out so that, the right-hand side is reduced to the Poisson brackets between $W^j(\theta,\phi)$ and $W_H^j(\theta,\phi)$ on the sphere [32].

In the case, when the Hamiltonian is an operator from the enveloping algebra of $su(2)$ invariant operators (and thus leaves invariant each $SU(2)$ irreducible subspace), its symbol does not depend on the angle ψ and the only admissible values of the index j in $W_H^j(\Omega)$ are integer. Besides, the Wigner function $W^j(\Theta)$ corresponding to different values of the index j evolve independently. Then, the above evolution equation is essentially simplified especially for the so-called semiclassical states, which are localized in each representation and 'among' representations (for instance, two-mode coherent states (35)):

$$\begin{aligned}\mathrm{i}\partial_t W^j(\Theta) = {}&\big(W_H^{j-i\partial_\psi}(\Omega) - W_H^{j+i\partial_\psi}(\Omega)\big)W^j(\Theta)\\ &+\mathrm{i}\varepsilon\left(\frac{1}{\sin\theta}\big(\partial_\phi^W\otimes\partial_\theta^H - \partial_\theta^W\otimes\partial_\phi^H\big) - \cot\theta\,\partial_\psi^W\otimes\partial_\theta^H\right)\\ &\times\left(W_H^{j-i\partial_\psi}(\Omega) + W_H^{j+i\partial_\psi}(\Omega)\right)W^j(\Theta)\end{aligned}\tag{45}$$

where $\Omega = (\phi,\theta)$ and the partial derivative on the angle ψ appearing in the index j of the Wigner function can be understood as a formal series on ∂_ψ:

$$W_H^{j+i\partial_\psi}(\Omega)W^j(\Theta) = W_H^j(\Omega)W^j(\Theta) + \mathrm{i}W_H^j(\Omega)\partial_\psi\left.\frac{\partial W^x(\Theta)}{\partial x}\right|_{x=j} + \cdots.$$

The first term in (45) shows that different invariant subspaces acquire different phases in the course of evolution. It is worth noting that the index j in the Wigner function $W^j(\Theta)$ may acquire both integer and semi-integer values.

As a simplest example let us consider evolution generated by the Hamiltonian

$$H = \omega J^2.$$

Taking into account (33) we immediately obtain the following approximate evolution equation:

$$\partial_t W^j(\Theta) \approx -\omega(j+1)\partial_\psi W^j(\Theta),$$

where j takes only integer values, which solution,

$$W^j(\Theta|t) = W^j(\Omega|\psi - (j+1)\omega t),$$

coincides with the corresponding exact solution.

It is worth noting the existence of another nontrivial term, $\cot\theta\,\partial_\theta W_H^j(\Omega)\partial_\psi W^j(\Theta)$, in (45) which directly relates the dynamics in each irreducible subspace with the evolution among different subspaces. As a second simple example we consider a linear but non-diagonal Hamiltonian

$$H = \omega S_x.$$

Taking into account (31) we easily obtain from (45) the following evolution equation:

$$\partial_t W^j(\Theta) = \left(-\frac{\cos\phi}{\sin\theta}\partial_\psi + \cot\theta\cos\phi\,\partial_\varphi + \sin\phi\,\partial_\theta\right) W^j(\Theta),$$

having a simple solution $W^j(\Theta|t) = W^j(\phi(t), \theta(t), \psi(t))$, where

$$\cos\theta(t) = \sqrt{1-c^2}\sin t + \cos\theta_0, \qquad c = \sin\theta_0\cos\phi_0,$$

$$\cos\phi(t) = \frac{c}{\sin\theta(t)}, \qquad \psi(t) = \psi_0 + c\int_0^t \frac{\mathrm{d}\tau}{1-\cos^2\theta(\tau)}.$$

A nontrivial contribution from $\psi(t)$ appears only if the initial state is distributed among several $SU(2)$ irreducible subspaces.

As a nontrivial example let us consider the following nonlinear two-mode Hamiltonian (Lipkin–Meshkov–Glick [39] model):

$$H = \frac{\chi}{2}((a^\dagger a)^2 + (b^\dagger b)^2) + \frac{g}{2}(a^\dagger b + ab^\dagger).$$

This Hamiltonian preserves the integral of motion $N = a^\dagger a + b^\dagger b$ corresponding to the excitation number conservation, so that it can be recasted in terms of the $su(2)$ algebra generators as follows:

$$H = gS_x + \chi S_z^2 + \chi N/2(N/2+1),$$

where $S_x = (a^\dagger b + ab^\dagger)/2$, $S_z = (a^\dagger a - b^\dagger b)/2$ and $S_x^2 + S_y^2 + S_z^2 = N/2(N/2+1)$. If the initial state belongs to a single $SU(2)$ irreducible subspace the standard semiclassical methods can be applied in the limit of large excitation numbers. Nevertheless, for an arbitrary initial state a generalized evolution equation (45) should be used. For instance, let us suppose that both fields are initially in strong coherent states: $|\alpha\rangle_a|\beta\rangle_b$. Then, the symbol of such a state (36) depends on ψ, so that the corresponding semiclassical evolution equation takes the form

$$\partial_t W^j(\Theta) = g\left(\cot\theta\cos\phi\,\partial_\phi + \sin\phi\,\partial_\theta - \frac{\cos\phi}{\sin\theta}\partial_\psi\right) W^j(\Theta)$$
$$- \chi(j+1)\left(\cos\theta\,\partial_\phi + \frac{2}{3}\partial_\psi\right) W^j(\Theta).$$

The general solution of the above equation is $W^j(\Theta|t) = W^j(\phi(t), \theta(t), \psi(t))$, where $\theta(t), \phi(t)$ are solutions of the corresponding classical equations of motion,

$$\dot{\theta} = -g\sin\phi,$$
$$\dot{\phi} = \chi(j+1)\cos\theta - g\cot\theta\cos\phi,$$

and the angle ψ evolves according to

$$\psi(t) = \psi_0 + \frac{2}{3}\chi(j+1)t + g\int_0^t \frac{\cos\phi(\tau)}{\sin\theta(\tau)}\,\mathrm{d}\tau. \tag{46}$$

This means, that to describe the dynamics of any operator non-diagonal on the $su(2)$ irreducible subspaces, for instance $a^\dagger + a$, in the semiclassical approach it is absolutely necessary to take into account time evolution of the angle ψ. Really, taking into account the representation of the annihilation operator a in the angular momentum basis $|J,m\rangle$, $a = \sum_{J,m}\sqrt{J+m}|J-1/2, m-1/2\rangle\langle J,m|$ we obtain its j-symbol

$$W_a^j(\Theta) = \sqrt{\frac{(2j+1)(2j+3)}{j+1}}\,\mathrm{e}^{-\mathrm{i}(\phi+\psi)/2}\cos\theta/2,$$

where j takes only semi-integer values. The symbol $W_a^j(\Theta)$ explicitly depends on the angle ψ, which means that equation (46) is vital for correct description of evolution of the average value $\langle a + a^\dagger\rangle$.

5. Conclusions

We introduced a Wigner-like mapping for systems with $SU(2)$ symmetry. Such mapping was obtained in the frame of the Stratonovich–Weyl approach and the principal requirement we have imposed is the covariance of symbols $W^j(\phi,\theta,\psi)$ under transformations from the $SU(2)$ group. The main advantage of the present approach is in the possibility to represent the whole density matrix, and not only its components in each $SU(2)$ irreducible subspace, in terms of c-valued functions. Also, such representation opens a possibility to study quasiclassical dynamics of quantum systems. In the particular case, when the Hamiltonian preserves irreducible subspaces the Wigner function evolves according to

$$W^j(\Theta|t) = W^j(\phi(t), \theta(t), \psi + g(\theta,\phi|t)),$$

where $\theta(t), \phi(t)$ are solutions of the classical equations of motion and the form of $g(\theta,\phi|t)$ depends on the Hamilton function.

We introduce a Wigner-like quasidistribution function to describe quantum systems with $SU(2)$ dynamic symmetry group. This function is defined in a three-dimensional group manifold and can be used to represent quantum states having components in several $SU(2)$ invariant subspaces. It particular, it allows us to 'draw' non-diagonal (between irreducible subspaces) elements of the density matrix.

An explicit differential Moyal-like form of the star product is found and analyzed in the semiclassical limit, which opens a possibility to study semiclassical dynamics of quantum systems including 'diffusion' among different $SU(2)$ irreducible subspaces. Such diffusion could appear only when the system's Hamiltonian mixes irreducible subspaces. Such effect can be observed, for instance, in the processes of interaction of polar molecules with electromagnetic fields [40] (the problem of alignment and orientation of molecules in external fields), when the interaction Hamiltonian has the following form:

$$H = J^2 + \omega\hat{z} + \alpha\hat{z}^2,$$

where $\hat{z}$ is defined in equation (34)). Another example where a specific mixture the $SU(2)$ invariant subspaces is produced during the evolution is the Lipkin–Meshkov–Glick model when one (or both) mode of the quantized field is pumped by an external source,

$$H = \frac{\chi}{2}((a^\dagger a)^2 + (b^\dagger b)^2) + \frac{g}{2}(a^\dagger b + ab^\dagger) + \chi(a + a^\dagger).$$

It is worth noting here that under the action of a and $a^\dagger$ the index j of the Wigner function $W^j(\Theta)$ is displaced on $1/2$, so that integer and semi-integer values of j are resulted to be connected in the course of the Hamiltonian evolution. Some other problems as dynamics of a charged particle in central field in the presence of electric field can also be studied in the frame of the present formalism. These problems will be discussed by separately elsewhere.

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Appendix A

In this appendix, we prove the properties of the kernel (19).

(a) Hermiticity

$$\begin{aligned}
\omega_j^\dagger(\vec{n}) &= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} D_{QQ'}^{K*}(\vec{n}) T_{KQ}^{\dagger\frac{j+Q'}{2}\frac{j-Q'}{2}} \\
&= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} D_{QQ'}^{K*}(\vec{n}) (-1)^{2K+Q'+Q} T_{K-Q}^{\frac{j-Q'}{2}\frac{j+Q'}{2}} \\
&= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} (-1)^{Q'-Q} D_{-Q-Q'}^{K}(\vec{n}) (-1)^{2K+Q'+Q} T_{K-Q}^{\frac{j-Q'}{2}\frac{j+Q'}{2}} \\
&= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} (-1)^{2K+2Q'} D_{QQ'}^{K}(\vec{n}) T_{KQ}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} = \omega_j(\vec{n}),
\end{aligned}$$

due to $K + Q'$ being always an integer.

(b) Covariance

$$\begin{aligned}
T_g \omega_j(\Theta) T_g^\dagger &= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} D_{QQ'}^{K}(\Theta) T_g T_{KQ}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} T_g^\dagger \\
&= \sum_{K=\{0,1/2\}}^{j} \sum_{Q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} D_{QQ'}^{K}(\Theta) \sum_q T_{Kq}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} D_{qQ}^{K}(g) \\
&= \sum_{K=\{0,1/2\}}^{j} \sum_{q,Q'=-K}^{K=\{0,1/2\}} \sqrt{\frac{2K+1}{j+1}} T_{Kq}^{\frac{j+Q'}{2}\frac{j-Q'}{2}} D_{qQ'}^{K}(g \cdot \Theta) = \omega_j(g \cdot \Theta).
\end{aligned}$$

(c) Integration of the kernel over the group (normalization)

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \int_0^{4\pi} \mathrm{d}\psi\ \omega_j(\Theta) = \sum_K^j \sqrt{\frac{2K+1}{j+1}} \sum_{Q,Q'=-K}^{K} 16\pi^2 \delta_{K,0}\delta_{Q,0}\delta_{Q',0} T_{KQ}^{\frac{j+Q'}{2}\ \frac{j-Q'}{2}}$$

$$= \begin{cases} \frac{16\pi^2}{\sqrt{j+1}} T_{00}^{j/2 j/2} = \frac{16\pi^2}{j+1} I_{j+1}, & j \text{ integer} \\ 0, & j \text{ semi-integer.} \end{cases}$$

(d) Trace of the kernel

$$\mathrm{tr}(\omega_j(\Theta)) = \sum_{K=\{0,1/2\}}^{j} \sqrt{\frac{2K+1}{j+1}} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta)\sqrt{j-Q'+1}\,\delta_{K0}\delta_{Q0}\delta_{j+Q'j-Q'}$$

$$= \begin{cases} 1, & j \text{ integer} \\ 0, & j \text{ semi-integer.} \end{cases}$$

(e) Reproductive kernel

$$\mathrm{tr}\big(\omega_j(\Theta)\omega_{j_1}^{\dagger}(\Theta')\big) = \sum_K^j \sqrt{\frac{2K+1}{j+1}} \sum_{K_1}^{j_1} \sqrt{\frac{2K_1+1}{j_1+1}} \sum_{Q,Q'=-K}^{K} \sum_{Q_1,Q_1'=-K_1}^{K_1}$$

$$\times D_{QQ'}^{K}(\Theta) D_{Q_1Q_1'}^{K_1*}(\Theta')\delta_{KK_1}\delta_{QQ_1}\delta_{j,j_1}\delta_{Q',Q_1'}$$

$$= \delta_{jj_1} \sum_K^j \frac{2K+1}{j+1} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta) D_{QQ'}^{K*}(\Theta') = \delta_{jj_1}\Delta_j(\Theta,\Theta'),$$

where $\Delta_j(\Theta,\Theta')$ is a delta function on the group manifold for the j-symbols,

$$\int \mathrm{d}V' \Delta_j(\Theta,\Theta') W_f^j(\Theta') = \int \mathrm{d}V' \sum_K^j \frac{2K+1}{j+1} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta) D_{QQ'}^{K*}(\Theta') W_f^j(\Theta')$$

$$= \int \mathrm{d}V' \sum_K^j \frac{2K+1}{j+1} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta) D_{QQ'}^{K*}(\Theta')$$

$$\times \sum_k^j \sum_{q,q'=-k}^{k} \sqrt{\frac{2k+1}{j+1}} f_{kq}^{\frac{j+q'}{2}\ \frac{j-q'}{2}} D_{qq'}^{k*}(\Theta')$$

$$= \sum_K^j \frac{2K+1}{j+1} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta) \sum_k^j \sum_{q,q'=-k}^{k} \sqrt{\frac{2k+1}{j+1}} f_{kq}^{\frac{j+q'}{2}\ \frac{j-q'}{2}}$$

$$\times \int \mathrm{d}V' D_{QQ'}^{K*}(\Theta') D_{qq'}^{k*}(\Theta')$$

$$= \sum_K^j \frac{2K+1}{j+1} \sum_{Q,Q'=-K}^{K} D_{QQ'}^{K}(\Theta) \sum_k^j \sum_{q,q'=-k}^{k}$$

$$\times \sqrt{\frac{2k+1}{j+1}} f_{kq}^{\frac{j+q'}{2}\ \frac{j-q'}{2}} \frac{16\pi^2}{2K+1}\delta_{Kk}\delta_{Qq}\delta_{Q'q'}$$

$$= \frac{16\pi^2}{j+1} \sum_k^j \sum_{q,q'=-k}^{k} D_{qq'}^{k}(\Theta) \sqrt{\frac{2k+1}{j+1}} f_{kq}^{\frac{j+q'}{2}\ \frac{j-q'}{2}} = \frac{16\pi^2}{j+1} W_f^j(\Theta).$$

Appendix B

In this appendix, we derive the general expression for the star-product operator.

First of all we note that the j-component, equation (18), of an operator can be expressed as

$$\hat{f} = \sum_{l,l'=0,1/2,1,\ldots} \sum_{k=|l'-l|}^{l'+l} \sum_{m=-k}^{k} f_{km}^{l'l} T_{km}^{l'l} \delta_{l'+l,j},$$

then the j-component of the product of two operators is represented as follows:

$$\hat{f}\hat{g} = \sum_{l_1,l_1'} \sum_{k_1,m_1'} \sum_{l_2,l_2'} \sum_{k_2,m_2'} f_{k_1m_1'}^{l_1'l_1} g_{k_2m_2'}^{l_2'l_2} T_{k_1m_1'}^{l_1'l_1} T_{k_2m_2'}^{l_2'l_2} \delta_{l_1'+l_2,j}. \tag{B.1}$$

Taking into account that a product of two tensor operators can be expanded as a linear combination of them,

$$T_{K_1Q_1}^{J_1'J_1} T_{K_2Q_2}^{J_2'J_2} = \delta_{J_1J_2'} \sqrt{(2K_1+1)(2K_2+1)} \sum_{K_3Q_3} (-1)^{2K_1+2K_2-K_3+J_2+J_1'} \times C_{K_1Q_1K_2Q_2}^{K_3Q_3} \begin{Bmatrix} K_1 K_2 K_3 \\ J_2 J_1' J_1 \end{Bmatrix} T_{K_3Q_3}^{J_1'J_2}$$

and that $6j$ can be conveniently represented in terms of the Clebsch–Gordan coefficients [32]

$$\begin{Bmatrix} K_1 K_2 K_3 \\ J_1 J_2 J_3 \end{Bmatrix} = \frac{(-1)^{K_1+J_2+J_3}}{\sqrt{2K_1+1}} \frac{F_{J_2J_3}^{K_1} F_{J_1J_3}^{K_2}}{F_{J_1J_2}^{K_3}} \sqrt{\frac{(K_1+J_2-J_3)!}{(K_1-J_2+J_3)!} \frac{(K_2+J_1-J_3)!}{(K_2-J_1+J_3)!}} \times \sum_{j=0}^{\infty} a_{J_3}^{j} b_{K_1K_2j}^{J_1J_2J_3} C_{K_2J_3-J_1+jK_3J_1-J_2}^{K_1J_3-J_2+j},$$

where

$$a_{J_3}^{j} = \frac{(-1)^j}{j!(2J_3+j+1)!},$$

$$b_{K_1K_2j}^{J_1J_2J_3} = \sqrt{\frac{(K_1-J_2+J_3+j)!(K_2-J_1+J_3+j)!}{(K_1+J_2-J_3-j)!(K_2+J_1-J_3-j)!}},$$

$$F_{J_1J_2}^{K} = \sqrt{(J_1+J_2+K+1)!(J_1+J_2-K)!},$$

we obtain for the j-symbol of the product (B.1)

$$\hat{f}\hat{g} = \sum_{l_1,l_1'} \sum_{k_1,m_1'} \sum_{l_2,l_2'} \sum_{k_2,m_2'} \sum_{l_3,l_3'} \sum_{k_3,m_3'} (-1)^{2k_1+2k_2-k_3+l_2+l_1'} f_{k_1m_1'}^{l_1'l_1} g_{k_2m_2'}^{l_2'l_2} \delta_{l_1'+l_2,j} \delta_{l_1,l_2'} \delta_{l_1',l_3'} \delta_{l_2,l_3} \tag{B.2}$$

$$\times \sqrt{(2k_1+1)(2k_2+1)} C_{k_1m_1'k_2m_2'}^{k_3m_3'} \begin{Bmatrix} k_1 k_2 k_3 \\ l_2 l_1' l_1 \end{Bmatrix} T_{k_3m_3'}^{l_3'l_3} \tag{B.3}$$

$$= \sum_{\bigcirc} (-1)^{2k_1+2k_2-k_3+l_2+l_1'} \sqrt{(2k_1+1)(2k_2+1)} f_{k_1m_1'}^{l_1'l_1} g_{k_2m_2'}^{l_2'l_2} \delta_{l_1'+l_2,j} \delta_{l_1,l_2'} \delta_{l_1',l_3'} \delta_{l_2,l_3} \tag{B.4}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^{k_1+l_1+l_1'}}{\sqrt{2k_1+1}} a_{l_1}^{n} b_{k_1k_2n}^{l_2l_1'l_1} \sqrt{\frac{(k_1-l_1+l_1')!}{(k_1+l_1-l_1')!} \frac{(k_2+l_2-l_2')!}{(k_2-l_2+l_2')!}} \tag{B.5}$$

$$\times \frac{F_{l_1'l_1}^{k_1} F_{l_2'l_2}^{k_2}}{F_{l_3'l_3}^{k_3}} C_{k_2\,l_2'-l_2+n\,k_3\,l_3-l_3'}^{k_1\,l_1-l_1'+n} C_{k_1m_1'k_2m_2'}^{k_3m_3'} T_{k_3m_3'}^{l_3'l_3}, \tag{B.6}$$

where $\sum_{\bigcirc} = \sum_{l_1,l_1'} \sum_{k_1,m_1'} \sum_{l_2,l_2'} \sum_{k_2,m_2'} \sum_{l_3,l_3'} \sum_{k_3,m_3'}$. Using the transformation relations for the Clebsch–Gordan coefficients

$$C^{k_3\, m_3'}_{k_1\, m_1'\, k_2\, m_2'} = (-1)^{k_1-m_1'}\sqrt{\frac{2k_3+1}{2k_2+1}}C^{k_2\, m_2'}_{k_3\, m_3'\, k_1 -m_1'},$$

$$C^{k_1\, l_1-l_1'+n}_{k_2\, l_2'-l_2+n\, k_3\, l_3-l_3'} = (-1)^{k_3+l_3-l_3'}\sqrt{\frac{2k_1+1}{2k_2+1}}C^{k_2\, l_2'-l_2+n}_{k_3\, l_3'-l_3\, k_1\, l_1-l_1'+n},$$

and the integral representation for a product of two Clebsch–Gordan coefficients

$$C^{k_2\, l_2'-l_2+n}_{k_3\, l_3'-l_3\, k_1\, l_1-l_1'+n} C^{k_2\, m_2'}_{k_3\, m_3'\, k_1 -m_1'} = (-1)^{l_1-l_1'+m_1'+n}\frac{2k_2+1}{16\pi^2}\int \mathrm{d}V\, D^{k_3}_{m_3'\, l_3'-l_3}(\Theta) D^{k_1 *}_{m_1'\, l_1'-l_1-n}(\Theta) D^{k_2 *}_{m_2'\, l_2'-l_2+n}(\Theta),$$

we rewrite (B.2) as follows:

$$\hat{f}\hat{g} = \sum_{\bigcirc}\frac{1}{16\pi^2}\sqrt{(2k_1+1)(2k_2+1)(2k_3+1)}\, f^{l_1' l_1}_{k_1 m_1'} g^{l_2' l_2}_{k_2 m_2'}\delta_{l_1'+l_2,j}\delta_{l_1,l_2'}\delta_{l_1',l_3'}\delta_{l_2,l_3}$$
$$\times \sum_{n=0}^{\infty}(-1)^n a^n_{l_1} b^{l_2 l_1' l_1}_{k_1 k_2 n}\sqrt{\frac{(k_1-l_1+l_1')!}{(k_1+l_1-l_1')!}\frac{(k_2+l_2-l_2')!}{(k_2-l_2+l_2')!}}$$
$$\times \frac{F^{k_1}_{l_1' l_1} F^{k_2}_{l_2' l_2}}{F^{k_3}_{l_3' l_3}}\int \mathrm{d}V\, D^{k_3}_{m_3'\, l_3'-l_3}(\Theta) D^{k_1 *}_{m_1'\, l_1'-l_1-n}(\Theta) D^{k_2 *}_{m_2'\, l_2'-l_2+n}(\Theta) T^{l_3' l_3}_{k_3 m_3'},$$

where the triangle rule, $|k_2-l_2| \leqslant l_1 \leqslant k_2+l_2$, has been taken into account to simplify the phase factor.

Now, we use the property of the Wigner D-functions

$$(J^+)^n D^{k_1 *}_{m_1'\, l_1'-l_1}(\Theta) = (-1)^n\sqrt{\frac{(k_1+l_1'-l_1)!}{(k_1+l_1'-l_1-n)!}\frac{(k_1-l_1'+l_1+n)!}{(k_1-l_1'+l_1)!}} D^{k_1 *}_{m_1'\, l_1'-l_1-n}(\Theta),$$

$$(J^-)^n D^{k_2 *}_{m_2'\, l_2'-l_2}(\Theta) = \sqrt{\frac{(k_2-l_2'+l_2)!}{(k_2-l_2'+l_2-n)!}\frac{(k_2+l_2'-l_2+n)!}{(k_2+l_2'-l_2)!}} D^{k_2 *}_{m_2'\, l_2'-l_2+n}(\Theta),$$

where

$$J^{\pm} = \mathrm{i}\mathrm{e}^{\mp \mathrm{i}\psi}\left[\pm\cot\theta\frac{\partial}{\partial\psi} + \mathrm{i}\frac{\partial}{\partial\theta} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right], \qquad J^0 = -\mathrm{i}\frac{\partial}{\partial\psi}$$

are the contravariant components of the $SU(2)$ generators in the rotating system [38], and obtain

$$\hat{f}\hat{g} = \sum_{\bigcirc}\frac{1}{16\pi^2}\sqrt{(2k_1+1)(2k_2+1)(2k_3+1)}\, f^{l_1' l_1}_{k_1 m_1'} g^{l_2' l_2}_{k_2 m_2'}\delta_{l_1'+l_2,j}\delta_{l_1,l_2'}\delta_{l_1',l_3'}\delta_{l_2,l_3} T^{l_3' l_3}_{k_3 m_3'}$$
$$\times \sum_{n=0}^{\infty} a^n_{l_1}\frac{F^{k_1}_{l_1' l_1} F^{k_2}_{l_2' l_2}}{F^{k_3}_{l_3' l_3}}\int \mathrm{d}V\left((J^+)^n D^{k_1 *}_{m_1'\, l_1'-l_1}(\Theta)\right)\left((J^-)^n D^{k_2 *}_{m_2'\, l_2'-l_2}(\Theta)\right) D^{k_3}_{m_3'\, l_3'-l_3}(\Theta).$$

After some long but straightforward algebra the above expression can be rewritten as follows:

$$\hat{f}\hat{g} = \sum_{j_1,j_2=0,1/2,\ldots}^{\infty} \frac{\sqrt{(j_1+1)(j_2+1)(j+1)}}{16\pi^2} \sum_{\oplus} \sqrt{\frac{(2k_1+1)(2k_2+1)(2k_3+1)}{(j_1+1)(j_2+1)(j+1)}} \tag{B.7}$$

$$\times\ f_{k_1 m_1'}^{\frac{j_1+m_1}{2}\,\frac{j_1-m_1}{2}} g_{k_2 m_2'}^{\frac{j_2+m_2}{2}\,\frac{j_2-m_2}{2}} \sum_{n=0}^{\infty} a^n_{\frac{j_1+j_2-j}{2}}$$

$$\times \int_0^{4\pi} \frac{\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_3}{(4\pi)^3}\,\mathrm{e}^{\mathrm{i}(j_1-j_2)\varphi_1}\,\mathrm{e}^{\mathrm{i}(j_1-j)\varphi_2}\,\mathrm{e}^{\mathrm{i}(j_2-j)\varphi_3} \tag{B.8}$$

$$\times \int \mathrm{d}V\,(J^+)^n\big(F_{j_1}\,\mathrm{e}^{-\mathrm{i}J^0(\varphi_1-\varphi_2)}D^{k_1*}_{m_1' m_1}(\Theta)\big)(J^-)^n\big(F_{j_2}\,\mathrm{e}^{-\mathrm{i}J^0(\varphi_1+\varphi_3)}D^{k_2*}_{m_2' m_2}(\Theta)\big) \tag{B.9}$$

$$\times \big(F_j^{-1}\,\mathrm{e}^{\mathrm{i}J^0(\varphi_2-\varphi_3)}D^{k_3}_{m_3' m_3}(\Theta)\big)T_{k_3 m_3'}^{\frac{j_3+m_3}{2}\,\frac{j_3-m_3}{2}}, \tag{B.10}$$

where the property $J^0 D^k_{m'm} = -m D^k_{m'm}$ and the integral representation of the Kroneker δ_{mn}-function (note that the indices can be semi-integer)

$$\delta_{mn} = \int_0^{4\pi} \frac{\mathrm{d}\phi}{4\pi}\,\mathrm{e}^{\mathrm{i}(m-n)\phi}$$

have been used, also the notation $\sum_{\oplus}$ means

$$\sum_{\oplus} = \sum_{k_1=\{0,1/2\}}^{j_1} \sum_{m_1,m_1'=-k_1}^{k_1} \sum_{k_2=\{0,1/2\}}^{j_2} \sum_{m_2,m_2'=-k_2}^{k_2} \sum_{k_3=\{0,1/2\}}^{j_3} \sum_{m_3,m_3'=-k_3}^{k_3},$$

and we have introduced the operator-valued function $F_j \equiv F_j(J^2)$, defined by its action on the D-functions, $F_j(J^2)D^k_{nm} = \sqrt{(k+j+1)!(j-k)!}D^k_{nm}$, where

$$J^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right]$$

is the Casimir operator in the rotating frame [38]. Note that by construction the value of $(j_1+j_2-j)/2$ is always integer.

In (B.7), the structure of the j-symbols is already recognizable so that using (21) and the self-conjugation property of the Casimir operator we obtain

$$W^j_{fg}(\Theta) = \sum_{j_1,j_2=0,1/2,\ldots}^{\infty} \sum_{n=0}^{\infty} a^n_{\frac{j_1+j_2-j}{2}} \sqrt{\frac{(j_1+1)(j_2+1)}{j+1}}$$

$$\times \int_0^{4\pi} \frac{\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\phi_3}{(4\pi)^5}\,\mathrm{e}^{\mathrm{i}(j_1-j_2)\phi_1}\,\mathrm{e}^{\mathrm{i}(j_1-j)\phi_2}\,\mathrm{e}^{\mathrm{i}(j_2-j)\phi_3} \tag{B.11}$$

$$\times\ F_j^{-1}\,\mathrm{e}^{-\mathrm{i}J^0(\phi_2-\phi_3)}\big[(J^+)^n\big(F_{j_1}\,\mathrm{e}^{-\mathrm{i}J^0(\phi_1-\phi_2)}W_f^{j_1}(\Theta)\big)(J^-)^n\big(F_{j_2}\,\mathrm{e}^{-\mathrm{i}J^0(\phi_1+\phi_3)}W_g^{j_2}(\Theta)\big)\big]. \tag{B.12}$$

Finally, taking into account $\exp(\alpha\partial/\partial x)f(x) = f(x+\alpha)$ and the commutation relations $[J^0, J^\pm] = \mp J^\pm$ we obtain after integrating on ϕ_3 the symbol of the product of $\hat{f}$ and $\hat{g}$ operators in terms of their individual symbols:

$$W^j_{fg}(\Theta) = \int_0^{4\pi} \frac{\mathrm{d}\varphi\,\mathrm{d}\varphi'}{(4\pi)^2} \sum_{n=0}^{\infty} \sum_{j_1,j_2=0,1/2,\ldots}^{\infty} a^n_{\frac{j_1+j_2-j}{2}} \sqrt{\frac{(j_1+1)(j_2+1)}{j+1}} F_j^{-1}(J^2)$$

$$\times\,[((J^+)^n F_{j_1}(J^2)\,\mathrm{e}^{\mathrm{i}(j_2-j+J^0)\varphi'}W_f^{j_1}(\Theta))((J^-)^n F_{j_2}(J^2)\,\mathrm{e}^{\mathrm{i}(j_1-j-J^0)\varphi}W_g^{j_2}(\Theta))].$$

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